Analysis 1, Summer 2023

List 10

Review for Exam 2

240. Calculate the following limits, if they exist:

(a) $\lim\limits_{x\to 4} \dfrac{x^2-x-12}{x^2-2x-8} = \boxed{\dfrac{7}{6}}$ from algebra or L'Hospital

(b) $\lim\limits_{x\to 4} \dfrac{x^2+x-12}{x^2-2x-8}$ $\boxed{\text{doesn't exist}}$

(c) $\lim\limits_{x\to \infty} xe^{-x} = \boxed{0}$ from L'Hospital with $\dfrac{x}{e^x}$

(d) $\lim\limits_{x\to 0^+} x^2\ln(x) = \boxed{0}$ from L'Hospiral with $\dfrac{x^2}{\ln(x)}$

(e) $\lim\limits_{x\to 1} x^2\ln(x) = \boxed{0}$ just from plugging in $x=1$

241. Compute $\lim\limits_{x\to 0} \dfrac{2e^x - x^2 - 2x - 2}{x^3}$. $\boxed{\dfrac{1}{3}}$

242. Compute $\lim\limits_{x\to 0} (\cos 6x)^{1/x^2}$. Hint: First compute $\lim\limits_{x\to 0} \ln\Big((\cos 6x)^{1/x^2}\Big)$.

$$\ln(a^b) = b\ln(a)$$

$$\ln\Big((\cos 6x)^{1/x^2}\Big) = \frac{1}{x^2}\ln(\cos 6x)$$

$$\begin{aligned}\lim_{x\to 0}\ln\Big((\cos 6x)^{1/x^2}\Big) &= \lim_{x\to 0}\frac{\ln(\cos 6x)}{x^2}\\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{1}{\cos 6x}\cdot(-6\sin(6x))}{2x}\\ &= \lim_{x\to 0}\frac{-3\sin(6x)}{x\cos(6x)}\\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-18\cos(6x)}{-6x\sin(x)+\cos(6x)}\\ &= \frac{-18}{0+1} = -18\end{aligned}$$

Since $\ln(\text{Answer}) = -18$, we have Answer $= \boxed{e^{-18}}$.

243. Give an equation for the tangent line to $y = \sqrt{x} + x^3$ at $x = 1$.

$\boxed{y = 2 + \tfrac{7}{2}(x-1)}$, or $y = \tfrac{7}{2}x - \tfrac{3}{2}$.

244. Use the Quotient Rule and the Product Rule to compute $\dfrac{dy}{dx}$ for $y = \dfrac{\ln(x)e^x}{x^2}$.

$$\frac{x^2(\ln(x)e^x + \frac{1}{x}e^x) - \ln(x)e^x(2x)}{x^4} = \boxed{\frac{e^x}{x^3}\big(x\ln(x) - 2\ln(x) + 1\big)}$$

245. Give an equation for the tangent line to $y = e^{4x\cos x}$ at $x = 0$. $\boxed{y = 4x+1}$

246. Calculate the derivative of e^{5x} in two ways:

(a) Use the rule $\frac{d}{dx}\left[e^x\right] = e^x$ along with the Chain Rule (here $e^{5x} = f(g(x))$ with $f(x) = e^x$ and $g(x) = 5x$). $e^{5x} \cdot 5$, which is $\boxed{5e^{5x}}$.

(b) Use algebra to rewrite $e^{5x} = (e^5)^x$ and then find the derivative of that function using the rule $\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln(a)$. $(e^5)^x \cdot \ln(e^5)$, which is $\boxed{5e^{5x}}$.

247. Calculate the derivative $\ln(5x)$ in two ways:

(a) Use the rule $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ along with the Chain Rule (here $\ln(5x) = f(g(x))$ with $f(x) = \ln(x)$ and $g(x) = 5x$). $\frac{1}{5x} \cdot 5$, which is $\boxed{\frac{1}{x}}$.

(b) Use algebra to rewrite $\ln(5x) = \ln(x) + \ln(5)$ and then find the derivative of that function. $\frac{1}{x} + 0$, which is $\boxed{\frac{1}{x}}$.

248. On what interval(s) is the function $x^3 - 6x + 11$ increasing? $\boxed{(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)}$

249. On what interval(s) is the function $x^3 - 6x + 11$ concave up? $\boxed{(0, \infty)}$

250. Find the x-coordinates of all critical points of $(2x + 3)e^{4x}$. $\boxed{x = -\frac{7}{4}}$

251. Find the x-coordinates of all inflection points of $x^4 + 9x^3 - 15x^2 + 17$. $\boxed{x = -5, x = \frac{1}{2}}$

252. Find the x-coordinates of all inflection points of $x^5 + 10x^4 - 50x^3 + 80x^2 - 15$. $\boxed{x = -8 \text{ only}}$

253. Find the absolute minimum of $f(x) = \frac{1}{4}x^4 - 4x^3 + 22x^2 - 48x + 32$ on $[1, 9]$. $\boxed{x = 2 \text{ AND } x = 6 \text{ both have } y = -4}$

254. Find the critical point(s) of $g(x) = \sqrt[3]{3x^2 + 4x + 1}$. $\boxed{x = -1, \quad x = -\dfrac{2}{3}, \quad x = -\dfrac{1}{3}}$

255. Find all the critical point(s) of the function

$$f(x) = x^4 - 12x^3 + 30x^2 - 28x$$

and classify each one as a local minimum, local maximum, or neither.

$\boxed{x = -1 \text{ is neither, } x = 7 \text{ is a local min}}$

256. Find all the critical point(s) of the function

$$f(x) = x(6 - x)^{2/3}$$

and classify each one as a local minimum, local maximum, or neither.

After simplifying, $f'(x) = \dfrac{18 - 5x}{3(6 - x)^{1/3}}$, so the critical points are $x = \frac{18}{5}$ (where f' is zero) and $x = 6$ (where f' doesn't exist). The fact that $\boxed{x = \frac{18}{5} \text{ is a local max}}$ can be found from the First or the Second Derivative Test, but the fact that $\boxed{x = 6 \text{ is a local min}}$ requires the First D. Test because $f''(6)$ is not defined.

☆257. Suppose $f(x)$ is a differentiable function for which $f(6) = 2$ and $f'(6) = 0$ and $f''(6) = 3$. Does the function have a local minimum at $x = 6$? $\boxed{\text{Yes}}$ because $f'(6) = 0$ and $f''(6) > 0$.
A local maximum? $\boxed{\text{No}}$ because that would require $f''(6) < 0$.

☆258. Suppose $f(x)$ is a differentiable function for which $f(3) = 0$ and $f'(3) = 2$ and $f''(3) = 6$. Does the function have a local minimum at $x = 3$? $\boxed{\text{No}}$ because $f'(3) \neq 0$ so $x = 3$ is not a critical point.
A local maximum? $\boxed{\text{No}}$ because $f'(3) \neq 0$ so $x = 3$ is not a critical point.

259. Calculate the value of $\displaystyle\int_{-2}^{2} (4 - x^2)\,\mathrm{d}x$. $\boxed{\dfrac{32}{3}}$

260. Find the value of $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\,\mathrm{d}x$. $\boxed{2\pi}$ This is the area of half of a disk with radius 2.

261. Compute the following indefinite integrals:
   (a) $\displaystyle\int 6\,\mathrm{d}x = \boxed{6x + C}$
   (b) $\displaystyle\int (2x + 6)\,\mathrm{d}x = \boxed{x^2 + 6x + C}$
   (c) $\displaystyle\int \frac{8}{x}\,\mathrm{d}x = \boxed{8\ln(x) + C}$
   (d) $\displaystyle\int \frac{8}{q}\,\mathrm{d}q = \boxed{8\ln(q) + C}$
   (e) $\displaystyle\int x^2 \cos(x^3)\,\mathrm{d}x = \boxed{\tfrac{1}{3}\sin(x^3) + C}$
   (f) $\displaystyle\int x^2 \cos(x)\,\mathrm{d}x = \boxed{2x\cos x + (x^2 - 2)\sin x + C}$ using parts twice.

262. Compute the following definite integrals:
   (a) $\displaystyle\int_1^5 (2x + 6)\,\mathrm{d}x = \boxed{48}$
   (b) $\displaystyle\int_0^{\pi} \tfrac{1}{3}\sin(u)\,\mathrm{d}u = \boxed{\dfrac{2}{3}}$
   (c) $\displaystyle\int_1^4 (x^3 + 2x - 7)\,\mathrm{d}x = \boxed{\dfrac{231}{4}}$
   (d) $\displaystyle\int_0^{\pi} 2e^t \sin(5t)\,\mathrm{d}t = \boxed{\dfrac{5}{13}(e^{\pi} + 1)}$

263. Compute the following integrals of rational functions:
   (a) $\displaystyle\int \frac{2x + 3}{10x^2 + 30x + 40}\,\mathrm{d}x = \boxed{\tfrac{1}{10}\ln(x^2 + 3x + 4) + C}$

(b) $\displaystyle\int \frac{10x^2+30x+40}{5x}\,dx = \boxed{x^2+6x+8\ln(x)+C}$

(c) $\displaystyle\int_1^3 \frac{10x^2+30x+40}{5x}\,dx = \boxed{20+8\ln(3)}$

(d) $\displaystyle\int \frac{3}{10x^2+40}\,dx = \boxed{\tfrac{3}{20}\arctan(\tfrac{x}{2})+C}$

(e) $\displaystyle\int_0^2 \frac{3}{10x^2+40}\,dx = \boxed{\frac{3\pi}{80}}$ because $\arctan(1)=\frac{\pi}{4}$ and $\arctan(0)=0$

(f) $\displaystyle\int_2^\infty \frac{1}{x^5}\,dx = \lim_{b\to\infty}\left(\frac{1}{64}-\frac{1}{4b^4}\right) = \boxed{\frac{1}{64}}$

264. Find the area of the domain

$$\{(x,y) : 0 \le x \le \pi,\ 0 \le y \le 5\sin(\tfrac{x}{2})\}.$$

$\displaystyle\int_0^\pi 5\sin(\tfrac{x}{2}) = \Big[-10\cos(\tfrac{x}{2})\Big]_{x=0}^{x=\pi} = \boxed{10}$

265. Find the area of the domain

$$\{(x,y) : 0 \le x \le \pi,\ 0 \le y \le 2x\sin(3x)+4x\}.$$

$\displaystyle\int_0^\pi (2x\sin(3x)+4x) = \underbrace{\int_0^\pi 2x\sin(3x)\,dx}_{\text{parts}} + \underbrace{\int_0^\pi 4x\,dx}_{\text{basic anti-derivative}} = \boxed{\frac{2}{3}\pi+2\pi^2}$

266. Find the area of the region bounded by the curves $y=x^2$ and $y=10-x^2$.

$\displaystyle\int_{-\sqrt5}^{\sqrt5} \left((10-x^2)-x^2\right)dx = \boxed{\frac{40\sqrt5}{3}}$

267. Calculate the area of the region bounded by $x=1$, $y=1$, and $y=\ln(x)$.

Option 1: $\displaystyle\int_1^e (\text{top}-\text{bottom})\,dx = \int_1^e (1-\ln(x))\,dx = \boxed{e-2}$

Option 2: $\displaystyle\int_0^1 (\text{right}-\text{left})\,dy = \int_0^1 (e^y-1)\,dy = \boxed{e-2}$

☆268. (a) Find the area of the region bounded by $y=x^2+a$ and $y=ax^2+2$, where $a\in[0,1)$ is a parameter (your answer will be a formula using a).

The intersections are when $x=\pm\sqrt{\frac{a-2}{a-1}}$. The area is

$$\int_{-\sqrt{(a-2)/(a-1)}}^{\sqrt{(a-2)/(a-1)}} \big((ax^2+2)-(x^2+a)\big)\,dx = \frac{-4(a-2)^{3/2}}{3(a-1)^{1/2}}.$$

(b) Among all such shapes, what is the smallest possible area? The function $f(a)=\frac{-4(a-2)^{3/2}}{3(a-1)^{1/2}}$ has $f'(a)=0$ when $a=\frac{1}{2}$.

269. Calculate the volume of the solid formed by rotating

$$\{(x,y) : 0 \le x \le \pi,\ 0 \le y \le x\sqrt{\sin x}\}$$

around the x-axis.

$\displaystyle\int_0^\pi \pi\left(x\sqrt{\sin x}\right)^2 dx = \pi\int_0^\pi x^2\sin x\,dx = \boxed{\pi^3-4\pi}$

270. Calculate the volume of the solid formed by rotating the region from Task 266 around the y-axis.

$2\int_0^5 \pi(\sqrt{y})^2\,\mathrm{d}y = \boxed{25\pi}$

271. Find the volume of the solid formed by rotating the region bounded by $y = -x^2 + 10x - 21$ and the x-axis around the x-axis.

The x-axis is $y = 0$. The solutions to $0 = -x^2 + 10x - 21$ are $x = 3$ and $x = 7$.

$V = \int_3^7 \pi(-x^2+10x-21)^2\,\mathrm{d}x = \pi\int_3^7 (x^4 - 20x^3 + 142x^2 - 420x + 441)\,\mathrm{d}x = \boxed{\frac{32}{3}}$

272. For the function

$$f(x) = xe^{-x/4},$$

(a) Give the equation for the tangent line to $y = f(x)$ at $x = -4$. $\boxed{y = 2ex + 4e}$

(b) Compute the limits $\lim_{x\to 1} f(x)$ $\boxed{e^{-1/4}}$ and $\lim_{x\to\infty} f(x)$. $\boxed{0}$

(c) Find the critical point of $f(x)$. (There is only one.) $\boxed{x = 4}$ and $y = 4/e$

(d) Is the point from part (c) a local minimum, local maximum, or neither? $\boxed{\text{local max}}$

(e) Find the inflection point of $f(x)$. (There is only one.) $\boxed{x = 8}$ and $y = 8/e^2$

(f) Calculate the area of the domain $\{(x, y) : 0 \le x \le 4, 0 \le y \le f(x)\}$.

$\int_0^4 f(x)\,\mathrm{d}x = \boxed{16 - \frac{32}{e}}$

(g) Calculate the volume of the solid formed by rotating the region from part (f) around the x-axis.

$\int_0^4 \pi f(x)^2\,\mathrm{d}x = \boxed{\left(16 - \frac{80}{e^2}\right)\pi}$